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On rotationally symmetric flow above an infinite rotating disk

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The similarity equations for rotationally symmetric flow above an infinite counter-rotating disk are investigated both numerically and analytically. Numerical solutions are found when α, the ratio of the disk's angular speed to that of the rigidly rotating fluid far from it, is greater than $-0{\cdot}68795$. It is deduced that there exists a critical value α_{cr} of α above which finite solutions are possible. The value of α_{cr} and the limiting structure as $\alpha \to \alpha_{cr}$ are found using the method of matched asymptotic expansions. The flow structure is found to consist of a thin viscous wall region above which lies a thick inviscid layer and yet another viscous transition layer. Furthermore, this structure is not unique: there can be any number of thick inviscid layers, each separated from the next by a viscous transition layer, before the outer boundary conditions on the solution are satisfied. However, comparison with the numerical solutions indicates that a single inviscid layer is preferred.

1. Introduction

The structure of the self-similar solutions of the steady incompressible Navier–Stokes equations for a rigidly rotating fluid bounded by an infinite rotating disk has been examined for special cases by von Kármán (1921), Bödewadt (1940), Batchelor (1951) and Rogers & Lance (1960). More recently Evans (1969) has obtained numerical solutions of the similarity equations for values of the parameter α, the ratio of the disk's angular speed to that of the rigidly rotating fluid far from it, in the ranges $-\infty < \alpha \leqslant -6{\cdot}211$ and $-0{\cdot}65 \leqslant \alpha \leqslant 0$. For values of α in the range $-6{\cdot}211 < \alpha < -0{\cdot}65$ the numerical method employed by Evans would not yield solutions unless suction was applied at the disk.

Ockendon (1972), using asymptotic methods, has found a complete first-order solution of the similarity equations for small values of a suction parameter when $-\infty < \alpha < -0{\cdot}6968$ which agrees well with the numerical results of Evans. Furthermore, her results confirm the non-existence of a finite solution without suction for $\alpha = -1$, which was rigorously proved by McLeod (1970). In addition it has been shown by Bodonyi (1973) that a steady solution is not possible as the

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large-time limit of the time-dependent solution of the unsteady similarity equations for $\alpha = -1$.

These results are especially interesting if one recalls that Evans could not obtain numerical solutions for $-6{\cdot}211 < \alpha < -0{\cdot}65$. Since the similarity equations have no solution when $\alpha = -1$, the question naturally arises as to the range of values of α for which this non-existence of solutions holds.

In this paper further numerical solutions of the similarity equations will be given, and the limiting value of α for which solutions exist will be deduced. In addition the structure of the flow field in the vicinity of the breakdown will be discussed.

2. Equations of motion

The governing differential equations for steady incompressible flow above an infinite rotating disk are readily obtained from the axisymmetric Navier–Stokes equations written in cylindrical co-ordinates (r, θ, z). If u, v and w represent the velocity components in the r, θ and z directions, respectively, then following von Kármán (1921), similarity solutions are sought in the non-dimensional form

$$\left.\begin{aligned} u(r,z) &= (\nu\Omega)^{\frac{1}{2}}xf'(y), \quad v(r,z) = (\nu\Omega)^{\frac{1}{2}}xg(y), \\ w(r,z) &= -2(\nu\Omega)^{\frac{1}{2}}f(y), \quad p(r,z) = \rho\nu\Omega(\tfrac{1}{2}x^2+h(y)), \\ r &= (\nu/\Omega)^{\frac{1}{2}}x, \quad z = (\nu/\Omega)^{\frac{1}{2}}y, \end{aligned}\right\} \tag{2.1}$$

where ν is the kinematic viscosity and ω and Ω are the angular speeds of the disk and fluid far from it, respectively. With this choice of variables the Navier–Stokes equations reduce to

$$f''' + 2ff'' - f'^2 + g^2 - 1 = 0, \tag{2.2}$$

$$g'' + 2fg' - 2gf' = 0, \tag{2.3}$$

$$h' + 2(f^2)' + 2f'' = 0, \tag{2.4}$$

where the primes denote differentiation with respect to y. It is also of interest to note that these equations are identical to those derived from boundary-layer theory, the reason being that the terms neglected by the usual boundary-layer arguments are identically zero for the choice of variables given in (2.1).

To complete the formulation of the boundary-value problem the appropriate boundary conditions must also be specified. At the disk the no-slip condition applies, so that $u = w = 0$ and $v = \omega r$. Far from the disk the fluid is assumed to be rigidly rotating with angular speed Ω. In view of the definitions (2.1) these conditions are equivalent to

$$f = f' = 0, \quad g = \alpha \quad \text{at} \quad y = 0, \tag{2.5}$$

$$f' \to 0, \quad g \to 1, \quad h \to 0 \quad \text{as} \quad y \to \infty, \tag{2.6}$$

where $\alpha \equiv \omega/\Omega$.

The pressure term $h(y)$ is uncoupled from the equations for f and g, thus f and g are first found from (2.2) and (2.3), and then h can be determined from

$$h(y) = 2(f^2(\infty) - f^2(y) - f'(y)). \tag{2.7}$$

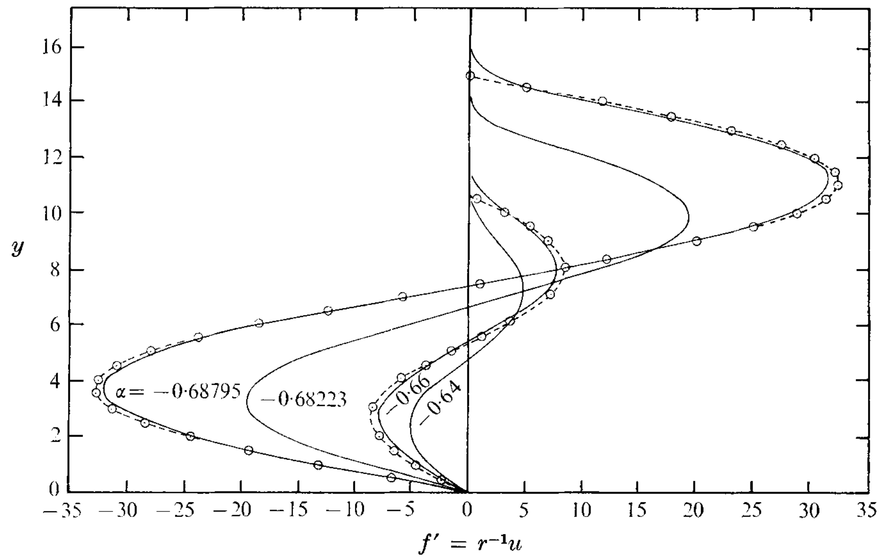

FIGURE 1. Transformed radial velocity for various α. ⊙, equation (4.23).

3. Numerical solutions

As mentioned previously, numerical solutions of the similarity equations for various values of the parameter α have been obtained by both Rogers & Lance (1960) and Evans (1969). For values of $\alpha \geqslant 0$ no difficulties were encountered in obtaining solutions. For a counter-rotating disk ($\alpha < 0$), however, Rogers & Lance were unable to find acceptable solutions, and Evans, using a shooting technique, was able to find solutions only when $\alpha \geqslant -0{\cdot}65$ and $\alpha \leqslant -6{\cdot}211$.

In order to study the behaviour of the similarity equations in the vicinity of $\alpha = -0{\cdot}65$ more closely further numerical integrations of the equations were carried out. The governing equations were written in their finite-difference form using centred differences, and the resulting nonlinear difference equations were solved recursively assuming initial approximations for f' and g. Only the results of the numerical computations will be presented here. A complete discussion of the numerical method is given by Bodonyi (1973).

Numerical solutions of (2.2) and (2.3) were found for values of α ranging between $+1{\cdot}0$ and $-0{\cdot}68795$. The results for $\alpha \geqslant 0$ agree well with those of Rogers & Lance and Evans and, therefore, will not be discussed further. With step sizes $\Delta y = 0{\cdot}05$ and $0{\cdot}135$ and the condition $y \rightarrow \infty$ approximated by $y = 20$, the solutions obtained for $-0{\cdot}65 \leqslant \alpha < 0$ agree favourably with those of Evans. In addition the method used in this study permitted new solutions for $-0{\cdot}68795 \leqslant \alpha \leqslant -0{\cdot}65$ to be found; and the resulting solutions clearly indicate that there is a critical value α_{cr} of α such that solutions exist for $\alpha > \alpha_{cr}$ but do not exist for $\alpha = \alpha_{cr}$. From the analysis to be discussed below it will be shown that this critical value α_{cr} is $-0{\cdot}6968$.

Figures 1 and 2 show the transformed radial and tangential velocity profiles for several values of α while figure 3 shows the behaviour of the transformed

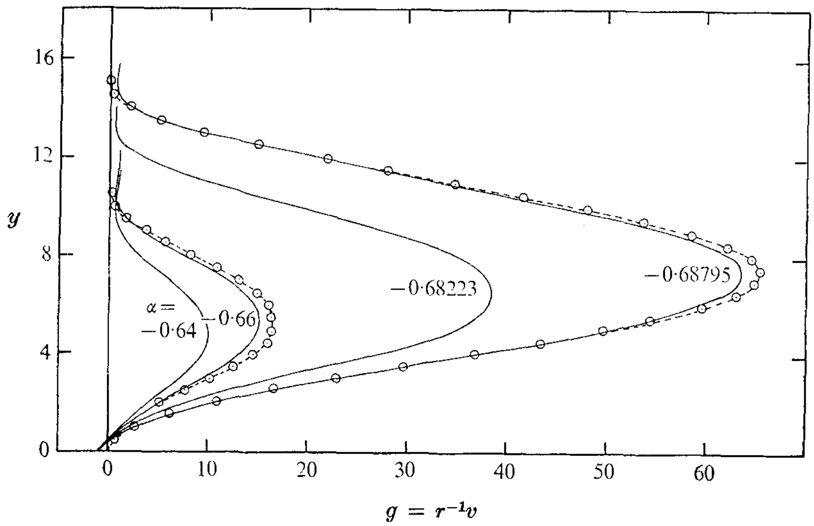

FIGURE 2. Transformed tangential velocity for various α. $\odot$, equation (4.24).

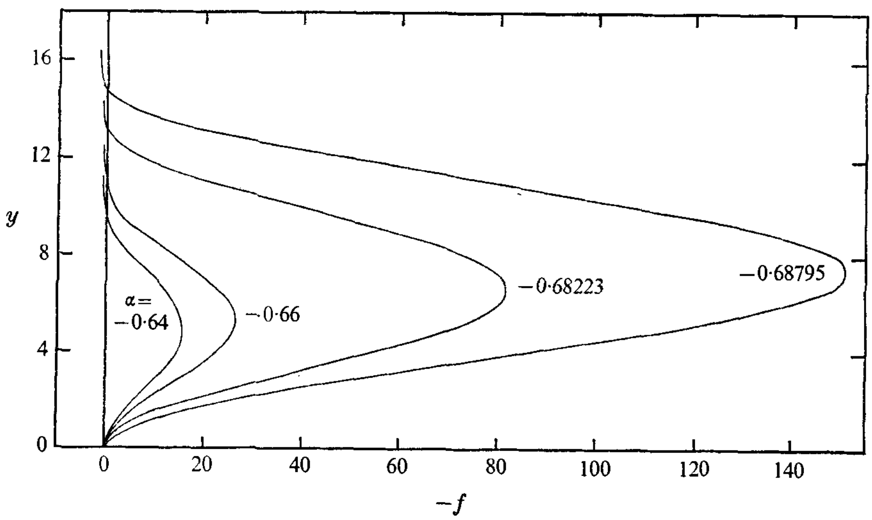

FIGURE 3. Transformed stream function for various α.

stream function for the same values of α. It may be clearly seen from these figures that the amplitudes of all the velocity components are becoming unbounded as $\alpha \to \alpha_{cr}$, suggesting that no finite solution exists for $\alpha = \alpha_{cr}$.

Further evidence for the breakdown of the similarity equations as $\alpha \to \alpha_{cr}$ is given in table 1 and figure 4, in which the radial and tangential wall shears are given as functions of α and were obtained by performing an h^2-extrapolation on the numerical solutions. The singular behaviour of the wall shears is apparent only for values of α quite close to α_{cr}, and even then only the radial wall shear strongly shows the presence of the singularity.

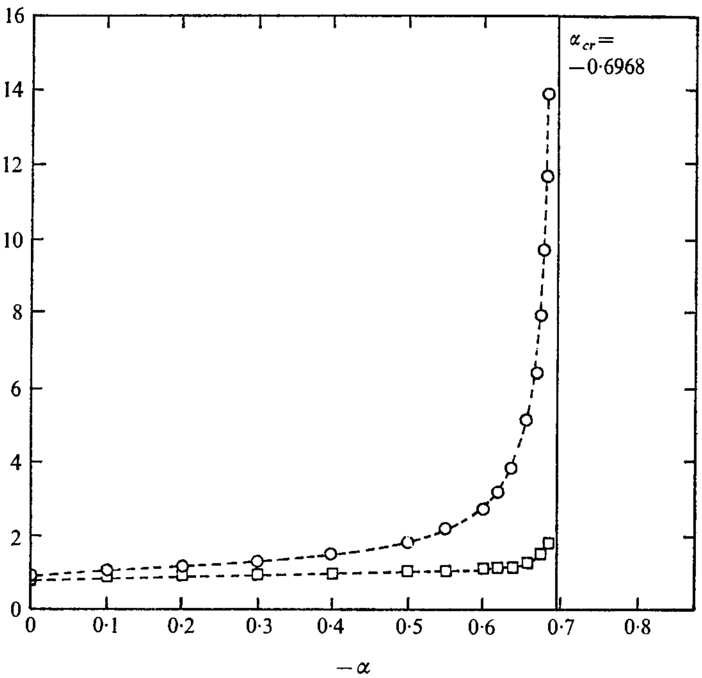

FIGURE 4. Transformed radial and tangential wall shears as functions of α. ○, $-u'(0;\alpha)$; □, $g'(0;\alpha)$.

4. Limiting structure as $\alpha \to \alpha_{cr}$

The singular behaviour of the numerical solutions as α approaches its critical value suggests that asymptotic methods for small values of the parameter $\alpha - \alpha_{cr}$ may be useful in deducing both the critical value of α and the limiting structure of the flow field. The asymptotic analysis to be discussed follows that used by Ockendon (1972) to study the flow above an infinite rotating disk when a small amount of suction is applied at the surface.

Inner region

Near the surface of the disk there is a thin layer of fluid where the viscous and inertia terms balance. To examine this region the normal co-ordinate y is stretched as follows:

$$\xi = \epsilon^{-\frac{1}{4}} y. \tag{4.1}$$

The dependent variables are scaled according to

$$f(y) = \epsilon^{-\frac{1}{4}}[f_0(\xi) + \epsilon f_1(\xi) + \ldots], \tag{4.2}$$

$$g(y) = g_0(\xi) + \epsilon g_1(\xi) + \ldots, \tag{4.3}$$

where

$$\epsilon = \alpha - \alpha_{cr} \tag{4.4}$$

and α_{cr} is to be found.

α	$f''(0)$	$g'(0)$	$f(\infty)$
0·00000	−0·93934	0·77139	−0·66939
−0·10000	−1·04031	0·82251	−0·74470
−0·20000	−1·15387	0·86903	−0·80318
−0·30000	−1·29182	0·91234	−0·82524
−0·40000	−1·48059	0·95622	−0·77346
−0·50000	−1·79243	1·01171	−0·58183
−0·55000	−2·16262	1·03259	−0·32917
−0·60000	−2·68333	1·10194	−0·05153
−0·64000	−3·81567	1·19964	0·33748
−0·65000	−4·33044	1·24438	0·45693
−0·66000	−5·08587	1·30532	0·59636
−0·66437	−5·54778	1·34000	0·66658
−0·67040	−6·39852	1·40000	0·77613
−0·67748	−7·97728	1·50000	0·93423
−0·68223	−9·77115	1·60000	1·07144
−0·68555	−11·79555	1·70000	1·19470
−0·68795	−14·06527	1·80000	1·30826

TABLE 1. Transformed radial and tangential wall shears and stream function as $y \to \infty$, as functions of α

Substituting (4.1)–(4.4) into (2.2)–(2.3) and equating like powers of ϵ yields the following set of differential equations:

$$f_0''' + 2f_0 f_0'' - f_0'^2 = 0, \tag{4.5}$$

$$g_0'' + 2f_0 g_0' - 2f_0' g_0 = 0, \tag{4.6}$$

$$f_1''' + 2f_0 f_1'' + 2f_0'' f_1 - 2f_0' f_1' = 1 - g_0^2, \tag{4.7}$$

$$g_1'' + 2f_0 g_1' - 2f_0' g_1 + 2f_1 g_0 - 2f_1' g_0 = 0. \tag{4.8}$$

The boundary conditions at $\xi = 0$ are given by

$$f_0 = f_0' = 0, \quad g_0 = \alpha_{\mathrm{cr}}, \tag{4.9}$$

$$f_1 = f_1' = 0, \quad g_1 = 1, \tag{4.10}$$

and as $\xi \to \infty$ it is required that the solutions do not become exponentially large.

The appropriate solution of (4.5) is found to be

$$f_0(\xi) = -\tfrac{1}{2}A\xi^2, \tag{4.11}$$

where A is an arbitrary constant at this stage, although it has been shown by Ockenden (1972) that A must be positive if the boundary conditions on f and g as $y \to \infty$ are to be satisfied. Thus in what follows it is assumed that $A > 0$. Substituting (4.11) into (4.6) and making a change of variables leads to a confluent hypergeometric equation for $g_0(\xi)$, and hence the solution can be written as

$$g_0(\xi) = \alpha_{\mathrm{cr}}\left\{ {}_1F_1(-\tfrac{2}{3};\, \tfrac{2}{3};\, \tfrac{1}{3}A\xi^3) - 2 \times 3^{\frac{2}{3}} A^{\frac{1}{3}} \left[\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\right]^2 \xi\, {}_1F_1(-\tfrac{1}{3};\, \tfrac{4}{3};\, \tfrac{1}{3}A\xi^3)\right\}, \tag{4.12}$$

where ${}_1F_1(a; b; t)$ is the confluent hypergeometric function.

Using the results for f_0 and g_0 given above it can be shown (see Bodonyi 1973) that the appropriate solution for $f_1(\xi)$ is given by

$$f_1(\xi) = c\xi^2 - \xi\int_0^\xi d\gamma \int_0^\gamma \eta^{-3}\, e^{\frac{1}{3}A\eta^3}\, d\eta \int_0^\eta x^2(g_0^2-1)\, e^{-\frac{1}{3}Ax^3}\, dx, \tag{4.13}$$

where c is an arbitrary integration constant at this stage. In particular it is found from (4.13) that

$$f_1'''(\xi) = e^{\frac{1}{3}A\xi^3}\left\{1 - A\int_0^\xi x^2 g_0^2\, e^{-\frac{1}{3}Ax^3}\, dx\right\} - g_0^2(\xi). \tag{4.14}$$

This result shows that $f_1(\xi)$ will become exponentially large as $\xi \to \infty$ unless

$$\int_0^\infty x^2 g_0^2\, e^{-\frac{1}{3}Ax^3}\, dx = \frac{1}{A} \tag{4.15}$$

or, using the change of variables

$$t = \tfrac{1}{3}Ax^3, \quad g_0(x) = \alpha_{cr}\bar{g}_0(t),$$

$$\int_0^\infty \bar{g}_0^2 e^{-t}\, dt = \alpha_{cr}^{-2}. \tag{4.16}$$

The integral appearing in (4.16) has been evaluated numerically, and its value is 2·0596. Hence

$$\alpha_{cr} = -0{\cdot}69680\ldots.$$

This number defines the lowest value of α above which finite solutions of the similarity equations can always be found. It is not surprising, therefore, that numerical solutions could not be found for some values of α less than $-0{\cdot}68795$.

Intermediate region

The expansions in the inner region fail when $\xi = O(\epsilon^{-\frac{1}{3}})$ and the variables must be rescaled in order to study the intermediate region in detail. The appropriate scalings are given by

$$y = \epsilon^{-\frac{1}{3}}\eta, \tag{4.17}$$

$$f(y) = \epsilon^{-\frac{2}{3}}\bar{f}_0(\eta) + \text{higher-order terms,}\dagger \tag{4.18}$$

$$g(y) = \epsilon^{-1}\bar{g}_0(\eta) + \text{higher-order terms.} \tag{4.19}$$

Substituting these expressions into (2.2) and (2.3) and equating like powers of ϵ to zero, it is found that to first order the equations are inviscid, i.e.

$$2\bar{f}_0\bar{f}_0'' - \bar{f}_0'^2 + \bar{g}_0^2 = 0, \tag{4.20}$$

$$\bar{f}_0\bar{g}_0' - \bar{f}_0'\bar{g}_0 = 0. \tag{4.21}$$

The boundary conditions on $\bar{f}_0$ and $\bar{g}_0$ as $\eta \to 0$ are found by matching the functions with the solutions in the inner region as the inner variable $\xi \to \infty$. Thus

$$\bar{f}_0 \sim -\tfrac{1}{2}A\eta^2, \quad \bar{f}_0' \sim -A\eta, \quad \bar{g}_0 \sim \beta_0\eta^2 \quad \text{as} \quad \eta \to 0, \tag{4.22}$$

† A referee has pointed out that the appropriate form of the expansion for higher-order terms is

$$\epsilon^{-\frac{1}{3}}\bar{f}_1(\eta) + \epsilon^{\frac{1}{3}}\log(\epsilon)\bar{f}_2(\eta) + \epsilon^{\frac{1}{3}}\bar{f}_3(\eta) + \ldots.$$

where
$$\beta_0 = -\alpha_{cr}\, 3^{\frac{1}{3}} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} A^{\frac{2}{3}}.$$

The solutions of (4.20) and (4.21) satisfying (4.22) are easily shown to be
$$\bar{f}_0(\eta) = -\lambda^2 A[1-\cos(\eta/\lambda)], \tag{4.23}$$
$$\bar{g}_0(\eta) = -\lambda A[1-\cos(\eta/\lambda)], \tag{4.24}$$
where
$$\lambda = -A/2\beta_0 = (0{\cdot}6857/\alpha_{cr}) A^{\frac{1}{3}}. \tag{4.25}$$

Outer region

The expansions fail in the intermediate region when $\eta/\lambda \to 2\pi$ because in this limit the functions $\bar{f}_0(\eta)$ and $\bar{g}_0(\eta)$ are both zero. In order to continue the solution further the η co-ordinate must be stretched in the neighbourhood of $\eta = 2\lambda\pi$, where there will be another viscous layer. The appropriate scalings for the variables are
$$y = \epsilon^{-\frac{1}{2}} 2\lambda\pi + \epsilon^{\frac{1}{2}}\zeta, \tag{4.26}$$
$$f(y) = \epsilon^{-\frac{1}{2}}\{\hat{f}_0(\zeta) + \epsilon\hat{f}_1(\zeta) + \ldots\}, \tag{4.27}$$
$$g(y) = \hat{g}_0(\zeta) + \epsilon\hat{g}_1(\zeta) + \ldots. \tag{4.28}$$

The differential equations for this region are identical to those for the inner region, i.e.
$$\hat{f}_0''' + 2\hat{f}_0\hat{f}_0'' - \hat{f}_0'^2 = 0, \tag{4.29}$$
$$\hat{g}_0'' + 2\hat{f}_0\hat{g}_0' - 2\hat{f}_0'\hat{g}_0 = 0. \tag{4.30}$$

The boundary conditions for $\zeta \to -\infty$ are found by matching with the intermediate region as $\eta \to 2\lambda\pi$. The results are
$$\hat{f}_0 \sim -\tfrac{1}{2}A\zeta^2, \quad \hat{f}_0' \sim -A\zeta, \quad \hat{g}_0 \sim \tfrac{1}{2}(A/\lambda)\zeta^2 \quad \text{as} \quad \zeta \to -\infty. \tag{4.31}$$

A complete discussion of the boundary conditions on $\hat{f}_0$ as $\zeta \to \infty$ is given by Ockendon (1972). It will suffice here to say that there are three possible forms for the outer boundary condition, two of which are physically acceptable. Therefore the behaviour of the limiting solution as $\alpha \to \alpha_{cr}$ is not unique. In fact it is possible to find solutions containing any number of the thick inviscid regions discussed above, each bounded by a thin transition layer, before the outer boundary conditions on $f'(y)$ and $g(y)$ are satisfied.

In the numerical solutions given in figures 1–3 there is a single thick inviscid region followed by a viscous layer in which the solutions asymptotically approach their limiting values. Thus for the outer region the boundary-value problem to be solved is given by (4.29) and (4.30) with boundary conditions (4.31) as $\zeta \to -\infty$, while as $\zeta \to \infty$ the following conditions are used:
$$\hat{f}_0' \to 0, \quad \hat{g}_0 \to 1 \quad \text{as} \quad \zeta \to +\infty. \tag{4.32}$$

The properties of (4.29) subject to the above boundary conditions have been investigated in detail by Ockendon, who found that a solution exists and is unique apart from an arbitrary shift in ζ. The solution of (4.29) has also been studied both analytically and numerically by Kuiken (1971).

α	$A^*(\alpha)$
−0·66000	0·42732
−0·66437	0·42396
−0·67040	0·41907
−0·67748	0·41339
−0·68223	0·40977
−0·68555	0·40746
−0·68795	0·40584

TABLE 2. Variation of A^* with α from the asymptotic solution of the steady similarity equations as $\alpha \to \alpha_{cr}$

A comparison of the first-order asymptotic theory for the intermediate region with the numerical solutions can be made once the value of the constant A has been determined. In contrast to the problem solved by Ockendon (1972), it is not possible to deduce analytically the value of A using only first-order theory. It can be shown, however, that A is fixed by requiring that the third-order term $\bar{f}_2(\xi)$, in the inner viscous region, should not become exponentially large as the inner variable $\xi \to \infty$. But to determine A a complicated integral involving $g_0(\xi)$, $f_1(\xi)$ and $g_1(\xi)$ must be evaluated numerically.

For the purposes of this study it is felt that an adequate determination of A can be obtained by using the results of the numerical computations. From (4.2) and (4.11)

$$f''(0;\alpha) = -(\alpha-\alpha_{cr})^{-\frac{3}{4}}A + O[(\alpha-\alpha_{cr})^{\frac{1}{4}}]. \tag{4.33}$$

Thus A can be defined by

$$A = \lim_{\alpha\to\alpha_{cr}} A^*(\alpha), \tag{4.34}$$

where

$$A^*(\alpha) = -(\alpha-\alpha_{cr})^{\frac{3}{4}} f''(0;\alpha). \tag{4.35}$$

Using the numerical results for $f''(0;\alpha)$ given in table 1 the variation of A^* with α is easily found, and the results are given in table 2. By extrapolating the values of A^* as $\alpha \to \alpha_{cr}$ it can be deduced that $A \simeq 0·399$.

Comparisons of the asymptotic solutions for $f'(y;\alpha)$ and $g(y;\alpha)$ in the intermediate region with the numerical solutions are shown in figures 1 and 2. The agreement is quite good over the entire range of integration with the exception of the regions near the inner and outer boundaries, wherein the solutions must be adjusted in the viscous layers to accommodate the boundary conditions.

The results of this study have shown that no finite solution of the similarity equations exists when $\alpha = -0·6968$. In addition McLeod (1970) proved that no solution exists when $\alpha = -1$. Thus it seems that no solutions of the boundary-value problem discussed in this study are possible for $-1 \leqslant \alpha \leqslant -0·6968$. For $-\infty < \alpha \leqslant -6·211$ solutions with and without suction have been found numerically by Evans (1969). However, for $-6·211 < \alpha \leqslant -1$, Evans was able to find solutions only if an appropriate amount of suction was applied at the disk. Ockendon's (1972) asymptotic analysis confirmed these results for small values of the suction parameter, and in addition she found that solutions for non-zero suction were possible for $-1 < \alpha < -0{\cdot}6968$, although numerical solutions in this range of α have not yet been given.

The reasons for the breakdown of the numerical solutions without suction in the neighbourhood of $\alpha = -6{\cdot}211$ are still not understood. Evans' method failed to yield results owing to the excessive amounts of computer time needed, and the numerical method used in this study fails to converge for no apparent reason when $\alpha \simeq -6{\cdot}211$. Therefore, it appears from Ockendon's analysis and the present work that solutions of the steady similarity equations for an infinite rotating disk do not exist when α lies in the range $-6{\cdot}211 < \alpha < -0{\cdot}6968$. This non-existence, however, has been rigorously proved, by McLeod, only for the single case $\alpha = -1$.

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